SOLUTIONS TO SELECTED QUESTIONS IN HOMEWORK 9

MATH 241

19.3.11

Proof. There are two singularities, and they are both different from the center of the annulus, so we need to first do the partition of fractions. Let $\frac{1}{z(z-3)} = \frac{A}{z} + \frac{B}{z-3}$, we can solve out $A = -\frac{1}{3}, B = \frac{1}{3}$. So we are reduced to the Laurent expansions of $(-\frac{1}{3})\frac{1}{z}$ and $\frac{1}{3}\frac{1}{z-3}$.

For $\frac{1}{z}$, 0 is outside the annulus, so

$$\frac{1}{z} = \frac{1}{z-4+4} = \frac{1}{4}\frac{1}{1+\frac{z-4}{4}} = \frac{1}{4}\sum_{k=0}^{\infty}\frac{(-1)^k}{4^k}(z-4)^k$$

For $\frac{1}{z-3}$, 3 is inside the annulus, so

$$\frac{1}{z-3} = \frac{1}{z-4}\frac{z-4}{z-3} = \frac{1}{z-4}\frac{1}{1+\frac{1}{z-4}} = \frac{1}{z-4}\sum_{k=0}^{\infty}(-1)^k\frac{1}{(z-4)^k} = \sum_{k=0}^{\infty}(-1)^k\frac{1}{(z-4)^{k+1}}$$

So in sum,

$$\frac{1}{z(z-3)} = (-\frac{1}{3})\cdot\frac{1}{4}\sum_{k=0}^{\infty}\frac{(-1)^k}{4^k}(z-4)^k + \frac{1}{3}\cdot\sum_{k=0}^{\infty}(-1)^k\frac{1}{(z-4)^{k+1}} = \frac{1}{3}\sum_{k=0}^{\infty}(-1)^k\frac{1}{(z-4)^{k+1}} - \frac{1}{12}\sum_{k=0}^{\infty}\frac{(-1)^k}{4^k}(z-4)^k$$

□

19.3.23

Proof. $f(z) = \frac{1}{(z-2)}\frac{1}{(z-1)^3}$, the first factor $\frac{1}{z-2}$ is good since the center is just 2, so only need to deal with the second one $\frac{1}{(z-1)^3}$. The domain is $\{z|0 < |z-2| < 1\}$, so the singularity 1 is outside of the annulus, therefore

$$\frac{1}{(z-1)^3} = \frac{1}{(1+(z-2))^3} = (1+(z-2))^{-3}$$

Here, the first method is to apply the General Binomial Theorem we stated in class, we get

$$(1+(z-2))^{-3} = 1 + \frac{(-3)}{1!}(z-2) + \frac{(-3)(-4)}{2!}(z-2)^2 + \frac{(-3)(-4)(-5)}{3!}(z-2)^3 + \cdots$$

Therefore the entire Laurent expansion of $f(z)$ is

$$\frac{1}{z-2} + \frac{(-3)}{1!} + \frac{(-3)(-4)}{2!}(z-2) + \frac{(-3)(-4)(-5)}{3!}(z-2)^2 + \cdots$$

On the other hand, the second method is to notice $(1+(z-2))^{-3} = \frac{1}{2}((1+(z-2))^{-1})''$, and we know $(1+(z-2))^{-1} = \sum_{k=0}^{\infty}(-1)^k(z-2)^k$, so we can differentiate term by term twice, get

$$((1+(z-2))^{-1})'' = \sum_{k=2}^{\infty}(-1)^k k(k-1)(z-2)^{k-2}$$

, therefore

$$(1+(z-2))^{-3} = \frac{1}{2}\sum_{k=2}^{\infty}(-1)^k k(k-1)(z-2)^{k-2}$$

$$f(z) = \frac{1}{(z-2)}\frac{1}{(z-1)^3} = \frac{1}{2}\sum_{k=2}^{\infty}(-1)^k k(k-1)(z-2)^{k-3}$$

You can check the two answers actually agree. □

19.4.9

Proof.

$$z(1-\cos(z^2)) = 1-(1-\frac{1}{2!}(z^2)^2+\frac{1}{4!}(z^2)^4-\cdots) = \frac{1}{2!}(z^2)^2-\frac{1}{4!}(z^2)^4+\cdots$$

So $z=0$ is a zero of order 2. □

Spring 09, #9

Proof. $f(z) = \frac{e^z}{(z-1)^2}$, we want its Laurent expansion on $\{z|1<|z-2|<2\}$, so the denominator is already in the good form, we just need the Taylor expansion of e^z at $z=1$. Since it is not centered at 0, we cannot directly use the formula we know. Instead, we are going to use the derivatives calculation. In fact, all the derivatives of e^z is just itself, so their evaluations at $z=1$ are all e. Therefore $e^z = e+e(z-1)+\frac{e}{2!}(z-1)^2+\cdots$, so $\frac{e^z}{(z-1)^2} = \frac{e}{(z-1)^2}+\frac{e}{z-2}+\sum_{k=0}^{\infty}\frac{e}{(k+2)!}(z-2)^k$. In particular, $b_2 = e, b_1 = e, a_0 = \frac{e}{2}, a_1 = \frac{e}{6}, a_2 = \frac{e}{24}$, so $b_2b_1a_0a_1a_2 = \frac{e^5}{288}$. □

Fall 09, #4

Proof. Two singularities $-i$ is inside the annulus $\{z|1<|z|<2\}$, and $2i$ is outside. First take the partition of fractions,

$$\frac{1}{(z-2i)(z+i)} = \frac{A}{z-2i}+\frac{B}{z+i}$$

Solve out $A = -\frac{1}{4}i, B = \frac{1}{4}i$. So $\frac{1}{(z-2i)(z+i)} = -\frac{1}{4}i\frac{1}{z-2i}+\frac{1}{4}i\frac{B}{z+i}$ and reduce to the Laurent expansions of $\frac{1}{z-2i}$ and $\frac{1}{z+i}$.

For $\frac{1}{z-2i}$, because $2i$ is outside the annulus, we have

$$\frac{1}{z-2i} = \frac{1}{2i}\frac{1}{1-\frac{z}{2i}} = -\frac{1}{2}i\frac{1}{1+\frac{1}{2}iz} = -\frac{1}{2}i\sum_{k=0}^{\infty}(-1)^k(\frac{1}{2}i)^k z^k$$

For $\frac{1}{z+i}$, because $-i$ is inside the annulus, we have

$$\frac{1}{z+i} = \frac{1}{z}\frac{z}{z+i} = \frac{1}{z}\frac{1}{1+\frac{i}{z}} = \frac{1}{z}\sum_{k=0}^{\infty}(-1)^k i^k z^k = \frac{1}{z} + \sum_{k=1}^{\infty}(-1)^k i^k z^{k-1}$$

Therefore in sum we have

$$\begin{aligned}
\frac{1}{(z-2i)(z+i)} &= -\frac{1}{4}i(-\frac{1}{2}i\sum_{k=0}^{\infty}(-1)^k(\frac{1}{2}i)^k z^k) + \frac{1}{4}i(\frac{1}{z} + \sum_{k=1}^{\infty}(-1)^k i^k z^{k-1}) \\
&= \frac{1}{4}i\frac{1}{z} - \frac{1}{8}\sum_{k=0}^{\infty}(-1)^k(\frac{1}{2}i)^k z^k + \frac{1}{4}i\sum_{k=0}^{\infty}(-1)^{k+1}i^{k+1}z^k \\
&= \frac{1}{4}i\frac{1}{z} + \sum_{k=0}^{\infty}[(-\frac{1}{8})(-1)^k(\frac{1}{2}i)^k + \frac{1}{4}i(-1)^{k+1}i^{k+1}]z^k \\
&= \frac{1}{4}i\frac{1}{z} + \sum_{k=0}^{\infty}(-1)^{k+1}i^k(\frac{1}{2^{k+3}} - \frac{1}{4})z^k
\end{aligned}$$

□